Perturbation-Energy Approach for the Development of the Nonlinear Equations of Ship Motion

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A perturbation analysis of the nonlinear coupling between the pitch and roll modes is used to illustrate that an energy approach can be used to advantage in developing the nonlinear equations governing the motion of ships. It is shown that employing Taylor series expansions to determine the loads on the hull of a ship can lead to the physically unrealistic prediction of self-sustained oscillations, unless certain relationships among the nonlinear coefficients are satisfied. It is shown that the simplified equations of motion which result after imposing these relationships can be found directly from an energy formulation of the problem. The energy approach is used to develop the nonlinear equations governing the roll and pitch modes to third order and the equations governing motions having six degrees-of-freedom to second order.

I. Introduction

THE importance of considering the nonlinear equations of motion was clearly illustrated in two earlier papers by the authors.[1,2] It was shown that in resonant situations the nonlinear response bears no similarity to the linear response. This paper is concerned with the development of the nonlinear equations of motion.

The equations of motion contain nonlinear inertial and hydrodynamic terms. In contrast with the inertial terms, the determination of the hydrodynamic terms is currently a difficult problem. Attempting to simplify the problem, one usually assumes that these terms can be separated into two categories: forces and moments which are generated by the motion of the ship in calm water, and forces and moments generated by wave action. In the present paper, attention is focused on the motion of ships in calm water.

Following Abkowitz,[3] one can assume that these forces and moments are analytic functions of the orientation, position, and motion of the ship and then represent these functions by Taylor series about the equilibrium configuration. The coefficients appearing in these expansions have to be provided by other considerations. The number of nonzero coefficients is somewhat less than the number of terms in the complete Taylor series because of symmetry and the assumptions that there are no higher-order acceleration terms and no velocity-acceleration interactions.

After these considerations are taken into account, the forms of the expansions may still be unsatisfactory, or at least misleading, because the equations of motion admit unrealistic self-sustained oscillations as solutions, unless some additional relationships among the coefficients are satisfied. Unfortunately, a nonlinear analysis is needed to determine these relationships.

As an alternative to assuming Taylor expansions for the forces and moments, we propose an energy formulation in which the ship and the sea are regarded as a single dynamic system. With this approach, one only needs to assume expansions for the kinetic energy, the dissipation, and the potential energy. The possibility of unrealistic predictions is eliminated if the potential energy increases with every displacement from the equilibrium position and the kinetic energy and the dissipation are positive definite for every motion. Of course, all three must also exhibit the proper symmetry.

In the present paper, we illustrate by means of a relatively simple example how the Taylor-series approach can lead to the prediction of unrealistic results, unless certain restrictions are imposed on the coefficients. Next we show that for the same example the energy formulation leads directly to the proper form of the equations of motion. Finally, we demonstrate the ease with which the energy approach can be used by developing the equations of motion for two more complicated examples.

II. Description of the Motion and Lagrange Equations

We employ two rectangular Cartesian coordinate systems: one fixed in space and the other fixed in the ship with its origin at the mass center, as shown in Fig. 1. Initially the two systems coincide. During the motion, the mass center is displaced and the axes fixed in the ship are rotated. The displacement of the mass center is described by the vector **R**. The rotation is described by the Euler angles associated with the following sequence: 1) a yaw-like rotation about the initial position of the z axis through the angle ψ, 2) a pitch-like rotation about the new position of the y axis through the angle θ, and 3) a roll-like rotation about the final position of the x axis through the angle ϕ. Thus, the motion is described by the elements of the following column matrix:

$$\{q\} = \begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \phi \\ \theta \\ \psi \end{Bmatrix} \tag{1}$$

where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the components of **R** referred to the spatially fixed coordinate system.

From any standard reference on dynamics such as Whittaker[4] and Meirovitch,[5] Lagrangian equations of motion can be obtained as follows:

$$d/dt\{\partial T/\partial \dot{q}\} - \{\partial T/\partial q\} + \{\partial V/\partial q\} + \{\partial D/\partial \dot{q}\} = \{N\} \tag{2}$$

where the kinetic energy T and the dissipation D are functions of $\{q\}$ and $\{\dot{q}\}$; where the potential energy V is a function of $\{q\}$; and where $\{N\}$ contains the generalized forces and moments. The dot denotes differentiation with respect to time.

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Because the moments and products of inertia with respect to the spatially fixed coordinate system vary with time during the motion of the ship, it is a difficult task to determine T. This difficulty can be avoided by expressing T in terms of body-fixed coordinates. To this end, we introduce the following:

$$\{\Pi\} = \begin{Bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{Bmatrix} \tag{3}$$

where u, v, and w are the components of the velocity of the mass center and p, q, and r are the components of the angular velocity; both sets are referred to the coordinate system fixed in the ship. In terms of the components of $\{\Pi\}$, the kinetic energy of the ship has the following familiar form:

$$T = (1/2)m(u^2 + v^2 + w^2) + (1/2)(I_{xx}p^2 + I_{yy}q^2 + I_{zz}r^2) - I_{xz}pr \tag{4}$$

The column matrix $\{\Pi\}$ is related to the column matrix $\{\dot{q}\}$ as follows:

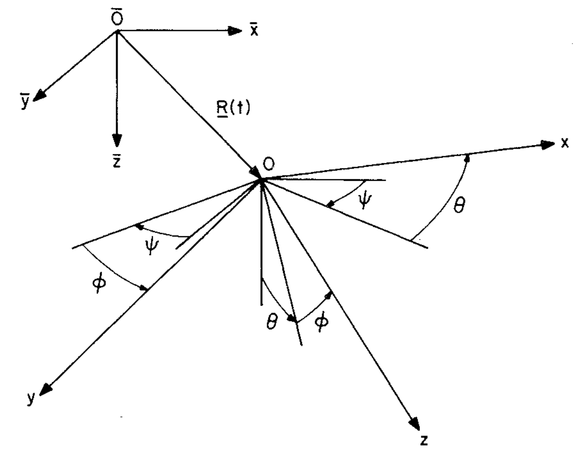

Fig. 1 Coordinate systems used to describe the motion $\bar{0}\bar{x}\bar{y}\bar{z}$ is fixed in space; $0xyz$ is fixed in the ship.

of quasi-coordinates. We take the second approach in the present paper.

$$\{\Pi\} = [\alpha]\,\{\dot{q}\} \tag{5a}$$

where

$$[\alpha] = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta & 0 & 0 & 0 \\ -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\theta\sin\phi & 0 & 0 & 0 \\ \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi & \cos\theta\cos\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\sin\theta \\ 0 & 0 & 0 & 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & 0 & 0 & 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Equation (5a) can be inverted to yield

$$\{\dot{q}\} = [\beta]\,\{\pi\} \tag{5b}$$

where

$$[\beta] = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi & 0 & 0 & 0 \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi & 0 & 0 & 0 \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & 0 & 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & 0 & 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{bmatrix}$$

We note that one cannot integrate the components of $\{\Pi\}$ directly and obtain physically meaningful coordinates. For this reason, these components are called the derivatives of quasi-coordinates.

At this point Eqs. (5) can be substituted into Eq. (4), similar changes in the expression for D can be made, and then the equations of motion from Eq. (2) can be obtained. The difficulty with this approach lies in determining the correct interpretation of the components of $\{N\}$. This difficulty can be avoided by substituting Eqs. (5) into Eq. (2) and obtaining Lagrangian equations in terms of quasi-coordinates. We take the second approach in the present paper.

After some manipulation with Eqs. (2) and (5), as outlined by Whittaker and Meirovitch, Lagrangian equations in terms of quasi-coordinates can be obtained as follows:

$$\frac{d}{dt}\left\{\frac{\partial \bar{T}}{\partial \Pi}\right\} + [\Gamma]\left\{\frac{\partial \bar{T}}{\partial \Pi}\right\} - [\beta]^T\left\{\frac{\partial \bar{T}}{\partial q} - \frac{\partial V}{\partial q}\right\} + \left\{\frac{\partial \bar{D}}{\partial \Pi}\right\} = \{Q\} \tag{6}$$

where $\bar{T}$ and $\bar{D}$ are now expressed in terms of $\{\Pi\}$ and $\{q\}$, $\{Q\}$ contains the generalized forces and moments with respect to the body-fixed axes, and

$$[\Gamma] = \begin{bmatrix} 0 & -r & q & 0 & 0 & 0 \\ r & 0 & -p & 0 & 0 & 0 \\ -q & p & 0 & 0 & 0 & 0 \\ 0 & -w & v & 0 & -r & q \\ w & 0 & -u & r & 0 & -p \\ -v & u & 0 & -q & p & 0 \end{bmatrix}$$

If we express the forces and moments acting on the ship in terms of Taylor series, then the terms involving V and $\bar{D}$ are lumped together, and an expansion is assumed for each equation of motion. In this case, $\bar{T}$ is the kinetic energy of the ship alone and Eq. (6) serves only to provide the inertial terms. However, we note that these are familiar terms and this procedure probably would not be followed if this were the only approach to be taken. On the other hand, if the sea and the ship together were regarded as a single dynamic system, then expansions for $\bar{T}$, $\bar{D}$, and V would need to be assumed. The forces and moments are obtained from the combination of derivatives indicated in Eq. (6). In both cases, $\{Q\}$ represents the generalized forces produced by the wave action and the control surfaces (both are assumed to be zero in the present paper).

III. Taylor-Series Approach

Substituting Eq. (4) into Eq. (6), evaluating the result for $u = v = w = r = o$, and retaining quadratic terms, we obtain

$$I_{xx}\dot{p} - I_{xz}pr = K \tag{7a}$$

$$I_{yy}\dot{q} + I_{xz}p^2 = M \tag{7b}$$

where K and M are the moments about the x and y axes, respectively.

Assuming K and M to be analytic functions of ϕ and θ and their derivatives, taking symmetry into account, and eliminating interactions between first and second derivatives as well as terms which are quadratic in the second derivatives, we obtain

$$K = K_\phi \phi + K_{\dot\phi}\dot\phi + K_{\ddot\phi}\ddot\phi + K_{\phi\theta}\phi\theta + K_{\phi\dot\theta}\phi\dot\theta + K_{\phi\ddot\theta}\phi\ddot\theta + K_{\theta\dot\phi}\theta\dot\phi + K_{\theta\ddot\phi\theta}\ddot\phi + K_{\dot\phi\dot\theta}\dot\phi\dot\theta + \text{cubic terms} \tag{8a}$$

and

$$M = M_\theta\theta + M_{\dot\theta}\dot\theta + M_{\ddot\theta}\ddot\theta + (1/2)M_{\phi\phi}\phi^2 + M_{\phi\dot\phi}\phi\dot\phi + (1/2)M_{\theta\theta}\theta^2 + M_{\theta\dot\theta}\theta\dot\theta + (1/2)M_{\dot\phi\dot\phi}\dot\phi^2 + (1/2)M_{\dot\theta\dot\theta}\dot\theta^2 + M_{\theta\ddot\theta}\theta\ddot\theta + \text{cubic terms} \tag{8b}$$

The coefficients K_ϕ, $K_{\dot\phi}$, M_θ, etc. (the stability derivatives) must be obtained from other considerations.

Substituting these expansions into Eqs. (7) and replacing p and q by

$$p = \dot\phi + \text{cubic terms}$$

and

$$q = \dot\theta + \text{cubic terms},$$

we obtain

$$\ddot\phi + \omega_1{}^2\phi = -\hat\mu_1\dot\phi + b_1\phi\theta + b_2\phi\dot\theta + b_3\phi\ddot\theta + b_4\theta\dot\phi + b_5\theta\ddot\phi + b_6\dot\theta\dot\phi \tag{9a}$$

and

$$\ddot\theta + \rho_2{}^2\theta = -\hat\mu_2\dot\theta + c_1\phi^2 + c_2\phi\dot\phi + c_3\phi\ddot\phi + c_4\theta^2 + c_5\theta\dot\theta + c_6\theta\ddot\theta + c_7\dot\phi^2 + c_8\dot\theta^2 \tag{9b}$$

where

$$[\omega_1{}^2, \hat\mu_1, b_1, b_2, b_3, b_4, b_5, b_6] = (I_{xx} - K_{\ddot\phi})^{-1}[-K_\phi, -K_{\dot\phi}, K_{\phi\theta}, K_{\phi\dot\theta}, K_{\phi\ddot\theta}, K_{\theta\dot\phi}, K_{\theta\ddot\phi}, K_{\dot\phi\dot\theta} + I_{xz}]$$

and

$$[\omega_2{}^2, \hat\mu_2, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8] = (I_{yy} - M_{\ddot\theta})^{-1}[-M_\theta, -M_{\dot\theta}, (1/2)M_{\phi\phi}, M_{\phi\dot\phi}, M_{\phi\ddot\phi}, (1/2)M_{\theta\theta}, M_{\theta\dot\theta}, M_{\theta\ddot\theta}, (1/2)M_{\dot\phi\dot\phi} - I_{xz}, (1/2)M_{\dot\theta\dot\theta}]$$

We seek an asymptotic expansion of the solution of Eqs. (9) which is valid for small, but finite, amplitudes. For convenience, let ϵ be a measure of the amplitude. Also, put $\hat\mu_1 = \epsilon\mu_1$ and $\hat\mu_2 = \epsilon\mu_2$.

According to the method of multiple scales,[6] we introduce different time scales defined as

$$T_n = \epsilon^n t \tag{10a}$$

The time derivatives are transformed as follows:

$$d/dt = D_0 + \epsilon D_1 + \ldots \tag{10b}$$

$$d^2/dt^2 = D_0^2 + 2\epsilon D_0 D_1 + \ldots \tag{10c}$$

where $D_n = \partial/\partial T_n$. Also, the functions ϕ and θ are assumed to have expansions of the form

$$\phi(t) = \epsilon\phi_1(T_0, T_1, T_2, \ldots) + \epsilon^2\phi_2(T_0, \ldots) + \ldots \tag{10d}$$

$$\theta(t) = \epsilon\theta_1(T_0, T_1, T_2, \ldots) + \epsilon^2\theta_2(T_0, \ldots) + \ldots \tag{10e}$$

Substituting Eqs. (10) into Eqs. (9) and equating coefficients of like powers of ϵ, we obtain

Order ϵ:

$$D_0{}^2\phi_1 + \omega_1{}^2\phi_1 = 0 \tag{11a}$$

$$D_0{}^2\theta_1 + \omega_2{}^2\theta_1 = 0 \tag{11b}$$

Order ϵ^2:

$$D_0{}^2\phi_2 + \omega_1{}^2\phi_2 = -2D_0D_1\phi_1 - \mu_1 D_0\phi_1 + b_1\phi_1\theta_1 + b_2\phi_1 D_0\theta_1 + b_3\phi_1 D_0{}^2\theta_1 + b_4\theta_1 D_0\phi_1 + b_5\theta_1 D_0{}^2\phi_1 + b_6(D_0\theta_1)(D_0\phi_1) \tag{12a}$$

$$D_0{}^2\theta_2 + \omega_2{}^2\theta_2 = -2D_0D_1\theta_1 - \mu_2 D_0\theta_1 + c_1\phi_1{}^2 + c_2\phi_1 D_0\phi_1 + c_3\phi_1 D_0{}^2\phi_1 + c_4\theta_1{}^2 + c_5\theta_1 D_0\theta_1 + c_6\theta_1 D_0{}^2\theta_1 + c_7(D_0\phi_1)^2 + c_8(D_0\theta_1)^2 \tag{12b}$$

We shall stop with one term in the expansions; consequently, the equations corresponding to higher orders of ϵ and all T_n for n greater than one are not needed.

Coupling between the two modes of oscillation occurs in the first approximation only if ω_2 is near $2\omega_1$. Thus, we consider this to be the case and express the nearness of ω_2 to $2\omega_1$ by introducing the detuning parameters σ such that

$$\omega_2 = 2\omega_1 + \hat\sigma, \hat\sigma = \epsilon\sigma \tag{13}$$

It is convenient to express the solutions to Eqs. (11) as

$$\phi = A_1(T_1)\exp(i\omega_1 T_0) + cc \tag{14a}$$

$$\theta = A_2(T_1)\exp(i\omega_2 T_0) + cc \tag{14b}$$

where cc represents the complex conjugate. At this point, A_1 and A_2 are unknown. They will be determined from the solvability conditions at the next level of approximation (i.e., by eliminating secular terms from the second-order problem).

Substituting Eqs. (13) and (14) into Eqs. (12), we obtain the second-order problem as

$$D_0{}^2\phi_2 + \omega_1{}^2\phi_2 = -i\omega_1(2A_1' + \mu_1 A_1)\exp(i\omega_1 T_0) + Z_1\bar{A}_1 A_2\exp[i(\omega_1 T_0 + \sigma T_1)] + cc + NST \tag{15a}$$

$$D_0^2\theta_2 + \omega_2^2\theta_2 = -i\omega_2(2A_2' + \mu_2 A_2)\exp(i\omega_2 T_0) + Z_2 A_1^2 \exp[i(\omega_2 T_0 - \sigma T_1)] + cc + NST \quad (15b)$$

where NST represents the terms which do not produce secular terms in the expansions and

$$Z_1 = b_1 - \omega_2^2 b_3 - \omega_1^2 b_5 + \omega_1\omega_2 b_6 + i(\omega_2 b_2 - \omega_1 b_4)$$

$$Z_2 = c_1 - (c_3 + c_7)\omega_1^2 + i\omega_1 c_2$$

The prime denotes differentiation with respect to T_1.

In order to eliminate the secular terms from ϕ_2 and θ_2, we set the coefficients of exp $(i\omega_1 T_0)$ and exp $(i\omega_2 T_0)$ equal to zero. This leads to the following solvability conditions:

$$-i\omega_1(2A_1' + \mu_1 A_1) + k_1\bar{A}_1 A_2 \exp[i(\sigma T_1 + \delta_1)] = 0 \quad (16a)$$

$$-i\omega_2(2A_2' + \mu_2 A_2) + k_2 A_1^2 \exp[-i(\sigma T_1 + \delta_2)] = 0 \quad (16b)$$

where

$$Z_1 = k_1 \exp(i\delta_1) \text{ and } Z_2 = k_2\exp(-i\delta_2) \quad (16c)$$

with positive k_n and real δ_n.

We let

$$A_n = (1/2)a_n \exp(i\beta_n) \quad (17)$$

with real a_n and β_n and define

$$\gamma = \beta_2 - 2\beta_1 + \sigma T_1 \quad (18)$$

Substituting these expressions into Eq. (16), separating the real and imaginary parts, and eliminating β_1 and β_2, we find

$$a_1' = -(1/2)\mu_1 a_1 + (k_1/4\omega_1)a_1 a_2 \sin(\gamma + \delta_1) \quad (19a)$$

$$a_2' = -(1/2)\mu_2 a_2 - (k_2/4\omega_2)a_1^2 \sin(\gamma + \delta_2) \quad (19b)$$

$$a_1 a_2 \gamma' = \sigma a_1 a_2 - (k_2/4\omega_2)a_1^3\cos(\gamma + \delta_2) + (k_1/2\omega_1)a_1 a_2^2 \cos(\gamma + \delta_1) \quad (19c)$$

The steady-state response is the solution of Eqs. (19) when a_1', a_2', and γ' are zero. In this case Eqs. (19a) and (19b) can be combined to give

$$[4\omega_1\omega_2\mu_1\mu_2 + k_1 k_2 a_1^2 \sin(\gamma + \delta_1)\sin(\gamma + \delta_2)]a_1 a_2 = 0 \quad (20)$$

A nontrivial solution can exist if, and only if,

$$\sin(\gamma + \delta_1)\sin(\gamma + \delta_2) < 0 \quad (21)$$

If a nontrivial solution exists, Eqs. (19a) and (19b) can be solved for a_2 and a_1^2/a_2. When these expressions are substituted into Eq. (19c), the result is

$$2\sigma + 2\mu_1\cot(\gamma + \delta_1) + \mu_2\cot(\gamma + \delta_2) = 0 \quad (22)$$

This equation is used to determine γ. We observe that ω_1, ω_2, and I_{xz}, and hence σ, δ_1, and δ_2 can be changed simply by changing the mass distribution for any given hull form. Thus, γ can be made to have any value. Consequently, from inequality (21), it follows that δ_1 must equal δ_2; otherwise, a nontrivial solution for a_1 and a_2 can exist.

Requiring δ_1 to equal δ_2 leads to

$$[K_{\phi\theta} - \omega_2^2 K_{\phi\dot\theta} - \omega_1^2 K_{\theta\ddot\phi} + \omega_1\omega_2(K_{\dot\phi\dot\theta} + I_{xz})]/(\omega_2 K_{\phi\dot\theta} - \omega_1 K_{\theta\dot\phi}) = [(1/2)M_{\phi\phi} - M_{\phi\ddot\phi} - (1/2)M_{\dot\phi\dot\phi} + I_{xz}]/\omega_1 M_{\phi\dot\phi} \quad (23a)$$

and

$$\frac{K_{\phi\theta} - \omega_2^2 K_{\phi\dot\theta} - \omega_1^2 K_{\theta\ddot\phi} + \omega_1\omega_2(K_{\dot\phi\dot\theta} + I_{xz})}{(1/2)M_{\phi\phi} - M_{\phi\ddot\phi} - (1/2)M_{\dot\phi\dot\phi} + I_{xz}} > 0 \quad (23b)$$

Cross multiplication of Eq. (23a) gives

$$(K_{\phi\dot\theta} + M_{\phi\dot\phi})I_{xz}\omega_1^2\omega_2 - K_{\theta\dot\phi}I_{xz}\omega_1^3 + [(1/2)M_{\dot\phi\dot\phi}K_{\theta\dot\phi} - K_{\theta\ddot\phi}M_{\phi\dot\phi})]\omega_1^3 - [(1/2)M_{\dot\phi\dot\phi}K_{\phi\dot\theta} + K_{\dot\phi\dot\theta}M_{\phi\dot\phi}]\omega_1^2\omega_2 - K_{\phi\theta}M_{\phi\dot\phi}\omega_1\omega_2^2 + (1/2)M_{\phi\phi}K_{\phi\dot\theta}\omega_2 + [K_{\phi\phi}M_{\phi\dot\phi} - (1/2)M_{\phi\phi}K_{\theta\dot\phi}]\omega_1 = 0 \quad (24)$$

We take the moment derivatives to be at most weak functions of ω_1 and ω_2 and hence consider them to be constants in the first approximation. Then by rearranging the mass distribution for any given hull form, we note that ω_1, ω_2, and I_{xz} will vary while the moment derivatives remain constant. Thus, Eq. (21) must hold for all values of ω_1, ω_2, and I_{xz}; and it follows that

$$K_{\phi\dot\theta} = 0 \quad (25a)$$

$$K_{\theta\dot\phi} = 0 \quad (25b)$$

$$M_{\phi\dot\phi} = 0 \quad (25c)$$

In order for inequality (23b) to hold for all values of ω_1, ω_2, and I_{xz}, it follows that

$$K_{\phi\theta} = M_{\phi\phi} \quad (26a)$$

$$K_{\theta\ddot\phi} + 4K_{\phi\ddot\theta} - 2M_{\phi\ddot\phi} = M_{\dot\phi\dot\phi} + 2K_{\dot\phi\dot\theta} \quad (26b)$$

where use was made of ω_2 being nearly twice ω_1. By performing an analysis, similar to that above, of the nonlinear coupling between the heave and pitch modes, it is found that

$$M_{\theta\dot\theta} = 0 \quad (27)$$

Imposing the conditions given in Eqs. (25) and (27) on Eqs. (7) and (8), we obtain the following set of reduced equations, subject to the conditions given in Eqs. (26):

$$I_{xx}\ddot\phi - I_{xz}\dot\phi\dot\theta = K_\phi\phi + K_{\dot\phi}\dot\phi + K_{\ddot\phi}\ddot\phi + K_{\phi\theta}\phi\theta + K_{\phi\dot\theta}\phi\dot\theta + K_{\theta\ddot\phi}\theta\ddot\phi + K_{\dot\phi\dot\theta}\dot\phi\dot\theta \quad (28a)$$

$$I_{yy}\ddot\theta + I_{xz}\dot\phi^2 = M_\theta\theta + M_{\dot\theta}\dot\theta + M_{\ddot\theta}\ddot\theta + (1/2)M_{\phi\phi}\phi^2 + M_{\phi\ddot\phi}\phi\ddot\phi + (1/2)M_{\theta\theta}\theta^2 + M_{\theta\ddot\theta}\theta\ddot\theta + (1/2)M_{\dot\phi\dot\phi}\dot\phi^2 + (1/2)M_{\dot\theta\dot\theta}\dot\theta^2 \quad (28b)$$

A brief summary follows: we began with a coupled pair of equations which contained 13 moment-derivative coefficients in the nonlinear terms. Then by using the method of multiple scales, we found that only 10 of these coefficients figure prominently in the solution when ω_1 is nearly twice ω_2; whereas, none figure prominently in the solution if the two frequencies are not commensurable. Finally, by eliminating the possibility of finite-amplitude, free oscillations persisting indefinitely in the presence of damping (i.e., self-sustained oscillations), we found that the number of independent coefficients which figure prominently in the solution is reduced to only 5. Consequently, it requires somewhat less to establish the prominent features of the nonlinear problem than was initially expected. In Sec. IV we show that the reduced equations, along with a further reduction, can be obtained by using an energy formulation.

Finally, we note that a preliminary result of this nature was found in Ref. 1. Moreover, there is an analog in the linear problem for one degree-of-freedom in which the motion decays only if the damping coefficient is positive (i.e., the damping force opposes the motion).

IV. Energy Formulation

Here we consider the ship and the sea to be a single dynamic system. Such an approach to the study of the motion of an object through an infinite, ideal liquid is discussed in detail by Lamb.[7] However, when the object is near an interface between two fluids, the formulation given by Lamb must be modified to 1) account for the variation in the kinetic energy with position and 2) include the potential energy and the dissipation due to waves at the interface. This modification is included in the present formulation.

The kinetic energy and the dissipation must be positive definite for every motion, and the potential energy must increase with every displacement from the undisturbed position. All three must account for the lateral symmetry of the ship. Consequently, we assume the following general forms:

$$\bar{T} = (1/2)(I_{xx} + I_1 + I_2\theta)p^2 + (1/2)(I_{yy} + I_3 + I_4\theta)q^2 + (1/2)(I_{zz} + I_5 + I_6\theta)r^2 + I_7\phi pq - (I_{xz} + I_8 + I_9\theta)pr + I_{10}\phi qr + \text{higher-order terms} \quad (29a)$$

$$V = (1/2)(V_1 + V_2\theta)\phi^2 + (1/2)(V_3 + V_4\theta)\theta^2 + \text{higher -order terms} \quad (29b)$$

$$\bar{D} = (1/2)(D_1p^2 + D_2q^2) + \text{higher-order terms} \quad (29c)$$

where all the quantities with numerical subscripts are constant coefficients which are obtained by other means. (These coefficients are the stability derivatives.) The remaining coefficients were defined in Sec. III.

Substituting Eqs. (29) into Eq. (6) and evaluating the result when $r = \dot{r} = 0$, we find

$$I_{xx}\ddot{\phi} - I_{xz}\dot{\phi}\dot{\theta} = -V_1\phi - D_1\dot{\phi} - I_1\ddot{\phi} - V_2\phi\theta - I_7\phi\ddot{\theta} - I_2\theta\ddot{\phi} + (I_8 - I_2)\dot{\theta}\dot{\phi} \quad (30a)$$

and

$$I_{yy}\ddot{\theta} + I_{xz}\dot{\phi}^2 = -V_3\theta - D_2\dot{\theta} - I_3\ddot{\theta} - (1/2)V_2\phi^2 - (3/2)V_4\theta^2 - I_7\phi\ddot{\phi} - I_4\theta\ddot{\theta} + [(1/2)I_2 - I_7 - I_8]\dot{\phi}^2 - (1/2)I_4\dot{\theta}^2 \quad (30b)$$

Comparing Eqs. (30) with Eqs. (28), one finds the following correspondence between the coefficients:

$$-V_2 = K_{\phi\theta} = M_{\phi\phi}, \quad (31a)$$

which is in accord with Eq. (26a),

$$-I_7 = K_{\phi\ddot{\theta}} = M_{\phi\ddot{\phi}}, \quad (31b)$$

$$-I_2 = K_{\theta\ddot{\phi}}, \quad (31c)$$

$$I_8 - I_2 = K_{\dot{\theta}\dot{\phi}} \quad (31d)$$

and

$$(1/2)I_2 - I_7 - I_8 = (1/2)M_{\dot{\phi}\dot{\phi}} \quad (31e)$$

Equations (31) can be manipulated to yield

$$K_{\theta\ddot{\phi}} + 2K_{\phi\ddot{\theta}} = M_{\dot{\phi}\dot{\phi}} + 2K_{\dot{\phi}\dot{\theta}} \quad (32)$$

which is in accord with Eq. (26b) after Eq. (31b) is used. The terms which would contain the coefficients corresponding to $K_{\theta\dot{\theta}}$, $K_{\theta\dot{\phi}}$, and $M_{\phi\dot{\phi}}$ do not appear in Eqs. (30). Finally, we note that the energy formulation clearly indicates that interactions between first and second derivatives as well as terms which are quadratic in the second derivatives do not appear in the equations of motion. This was assumed in Sec. III.

To determine the form of the equations through third-order terms, we only need to include the fourth-order nonlinear terms in the kinetic energy, potential energy, and dissipation. We readily arrive at the following general forms:

$$\bar{T} = (1/2)(I_{xx} + I_1 + I_2\theta + I_{11}\theta^2 + I_{12}\phi^2)p^2 + (1/2)(I_{yy} + I_3 + I_4\theta + I_{13}\theta^2 + I_{14}\phi^2)q^2 + (1/2)(I_{zz} + I_5 + I_6\theta + I_{15}\theta^2 + I_{16}\phi^2)r^2 + (I_7 + I_{17}\theta)\phi pq - (I_{xz} + I_8 + I_9\theta + I_{18}\theta^2 + I_{19}\phi^2)pr + (I_{10} + I_{20}\theta)\phi qr + \text{higher-order terms} \quad (33a)$$

$$V = (1/2)(V_1 + V_2\theta + V_5\theta^2 + V_6\phi^2)\phi^2 + (1/2)(V_3 + V_4\theta + V_7\theta^2)\theta^2 + \text{higher-order terms} \quad (33b)$$

$$\bar{D} = (1/2)(D_1 + D_3\theta^2 + D_4\phi^2)p^2 + (1/2)(D_2 + D_5\theta^2 + D_6\phi^2)q^2 + (1/4)(D_7p^4 + D_8q^4) + (1/2)D_9p^2q^2 + \text{higher-order terms} \quad (33c)$$

We note that eliminating the cubic terms in the expansion for the dissipation function makes the resulting equations consistent with the second-order equations already obtained, i.e., Eqs. (30). In this manner, we are guided by the perturbation analysis performed previously which led us to the correct form of the reduced equations.

Substituting Eqs. (33) into Eq. (6), expanding the trigonometric functions for small arguments, and evaluating the result at $r = \dot{r} = 0$, we obtain

$$I_{xx}\dot{p} - I_{xz}pq = -V_1\phi - D_1p - I_1\dot{p} - V_2\phi\theta - I_7\phi\dot{q} - I_2\theta\dot{p} - I_7\dot{\phi}q - I_2\dot{\theta}p + (I_7 + I_8)pq - V_5\phi\theta^2 - 2V_6\phi^3 - D_3\theta^2p - D_4\phi^2p - D_7p^3 - D_9pq^2 - 2I_{11}\theta\dot{\theta}p - I_{11}\theta^2\dot{p} - 2I_{12}\phi\dot{\phi}p - I_{12}\phi^2\dot{p} - I_{17}\theta\dot{\phi}q - I_{17}\theta\phi\dot{q} - I_{17}\phi\dot{\theta}q + I_{12}\phi p^2 + (I_{17} - I_9)\theta pq + (I_{14} - I_{10})\phi q^2 \quad (34a)$$

$$I_{yy}\dot{q} + I_{xz}p^2 = -V_3\theta - D_2q - I_3\dot{q} - (1/2)V_2\phi^2 - I_7\phi p - (3/2)V_4\theta^2 - I_4\theta\dot{q} - I_7\dot{\phi}p + [(1/2)I_2 - I_8]p^2 - I_4\dot{\theta}q + (1/2)I_4q^2 - [V_1 + V_5 - (1/2)V_3]\theta\phi^2 - 2V_7\theta^3 - D_5\theta^2q - D_6\phi^2q - D_8q^3 - D_9qp^2 - 2I_{13}\theta\dot{\theta}q - I_{13}\theta^2\dot{q} - 2I_{14}\phi\dot{\phi}q - I_{14}\phi^2\dot{q} - I_{17}\theta\phi\dot{p} - I_{17}\theta\dot{\phi}p - I_{17}\phi\dot{\theta}p + (I_9 + I_{11})\theta p^2 + I_{13}\theta q^2 + (I_{10} + I_{17})\phi pq \quad (34b)$$

We now consider a laterally symmetric ship which is free to oscillate with all six degrees-of-freedom. We write the kinetic energy of the ship and the fluid as

$$\bar{T} = (1/2)(m + m_1 + m_2z + m_3\theta)u^2 + (1/2)(m + m_4 + m_5z + m_6\theta)v^2 + (1/2)(m + m_7 + m_8z + m_9\theta)w^2 + (1/2)(I_{xx} + m_{10} + m_{11}z + m_{12}\theta)p^2 + (1/2)(I_{yy} + m_{13} + m_{14}z + m_{15}\theta)q^2 + (1/2)(I_{zz} + m_{16} + m_{17}z + m_{18}\theta)r^2 + m_{19}\phi uv + (m_{20} + m_{21}z + m_{22}\theta)uw + m_{23}\phi up + (m_{24} + m_{25}z + m_{26}\theta)uq + m_{27}\phi ur + m_{28}\phi vw + (m_{29} + m_{30}z + m_{31}\theta)vp + m_{32}\phi vq + (m_{33} + m_{34}z + m_{35}\theta)vr + m_{36}\phi wp + (m_{37} + m_{38}z + m_{39}\theta)wq + m_{40}\phi wr + m_{41}\phi pq + (-I_{xz} + m_{42} + m_{43}z + m_{44}\theta)pr + m_{45}\phi qr + \text{higher-order terms} \quad (35a)$$

We express the potential energy of the ship as

$$V = (1/2)(V_1 + V_2 z + V_3\theta)z^2 + (1/2)(V_4 + V_5 z + V_6\theta)\phi^2 + (1/2)(V_7 + V_8 z + V_9\theta)\theta^2 + V_{10}\theta z + \text{higher-order terms} \tag{35b}$$

and write the dissipation function as

$$\overline{D} = (1/2)(D_1u^2 + D_2v^2 + D_3w^2 + D_4p^2 + D_5q^2 + D_6r^2) + D_7uw + D_8uq + D_9vp + D_{10}vr + D_{11}wq + D_{12}pr + \text{higher-order terms} \tag{35c}$$

In Eq. (35a) m is the mass of the ship. All the coefficients with numerical subscripts are stability derivatives.

Substituting Eqs. (36) into Eqs. (6) and expanding the trigonometric functions for small arguments, we obtain the following equations:

$$(m + m_1)\dot{u} + m_{20}\dot{w} + m_{24}\dot{q} + D_1u + D_7w + D_8q = V_{10}\theta^2 + V_1z\theta - m_2z\dot{u} - m_2\dot{z}u - m_3\theta\dot{u} - m_3\dot{\theta}u - m_{19}\phi\dot{v} - m_{19}\dot{\phi}v - m_{21}z\dot{w} - m_{21}\dot{z}w - m_{22}\theta\dot{w} - m_{22}\dot{\theta}w - m_{23}\phi\dot{p} - m_{23}\dot{\phi}p - m_{25}z\dot{q} - m_{25}\dot{z}q - m_{26}\theta\dot{q} - m_{26}\dot{\theta}q - m_{27}\phi\dot{r} - m_{27}\dot{\phi}r + (m + m_4)vr + m_{29}pr - (m + m_7)wq - m_{20}uq - m_{37}q^2 + m_{33}r^2 \tag{36a}$$

$$(m + m_4)\dot{v} + m_{29}\dot{p} + m_{33}\dot{r} + D_2v + D_9p + D_{10}r = -V_1z\phi - V_{10}\phi\theta - m_5z\dot{v} - m_5\dot{z}v - m_6\theta\dot{v} - m_6\dot{\theta}v - m_{19}\phi\dot{u} - m_{19}\dot{\phi}u - m_{28}\phi\dot{w} - m_{28}\dot{\phi}w - m_{30}z\dot{p} - m_{30}\dot{z}p - m_{31}\theta\dot{p} - m_{31}\dot{\theta}p - m_{32}\phi\dot{q} - m_{32}\dot{\phi}q - m_{34}z\dot{r} - m_{34}\dot{z}r - m_{35}\theta\dot{r} - m_{35}\dot{\theta}r - (m + m_1)ur - m_{20}wr - m_{24}qr + (m + m_7)wp + m_{20}up + m_{37}pq \tag{36b}$$

$$(m + m_7)\dot{w} + m_{20}\dot{u} + m_{37}\dot{q} + D_3w + D_7u + D_{11}q + V_1z + V_{10} = -(3/2)V_2z^2 - (1/2)V_5\phi^2 - (1/2)V_8\theta^2 - V_3z\theta - m_8z\dot{w} - m_8\dot{z}w - m_9\theta\dot{w} - m_9\dot{\theta}w - m_{21}z\dot{u} - m_{21}\dot{z}u - m_{22}\theta\dot{u} - m_{22}\dot{\theta}u - m_{28}\phi\dot{v} - m_{28}\dot{\phi}v - m_{36}\phi\dot{p} - m_{36}\dot{\phi}p - m_{38}z\dot{q} - m_{38}\dot{z}q - m_{39}\theta\dot{q} - m_{39}\dot{\theta}q - m_{40}\phi\dot{r} - m_{40}\dot{\phi}r + (m + m_1 + m_{25})uq - (m + m_4 - m_{30})vp - (m_{33} - m_{43})pr + m_{21}uw + (m_{20} + m_{38})wq + m_{34}vr + (1/2)m_2u^2 + (1/2)m_2v^2 + (1/2)m_8w^2 + [(1/2)m_{11} - m_{29}]p^2 + [(1/2)m_{14} + m_{24}]q^2 + (1/2)m_{17}r^2 \tag{36c}$$

$$(I_{xx} + m_{10})\dot{p} + m_{29}\dot{v} + (m_{42} - I_{xz})\dot{r} + D_4p + D_9v + D_{12}r + V_4\phi = -V_5z\phi - V_6\phi\theta - m_{11}z\dot{p} - m_{11}\dot{z}p - m_{12}\theta\dot{p} - m_{12}\dot{\theta}p - m_{23}\phi\dot{u} - m_{23}\dot{\phi}u - m_{30}z\dot{v} - m_{30}\dot{z}v - m_{31}\theta\dot{v} - m_{31}\dot{\theta}v - m_{36}\phi\dot{w} - m_{36}\dot{\phi}w - m_{41}\phi\dot{q} - m_{41}\dot{\phi}q - m_{43}z\dot{r} - m_{43}\dot{z}r - m_{44}\theta\dot{r} - m_{44}\dot{\theta}r + (m_4 - m_7 + m_{28})vw + (m_{29} + m_{36})wp + (m_{33} + m_{37} + m_{40})wr + (m_{24} + m_{27})ur + (m_{41} - m_{42} + I_{xz})pq + (m_{19} - m_{20})uv + m_{23}up + (m_{32} - m_{33} - m_{37})vq + (m_{13} - m_{16} + m_{45} + I_{yy})qr \tag{36d}$$

$$(I_{yy} + m_{13})\dot{q} + m_{24}\dot{u} + m_{37}\dot{w} + D_5q + D_8u + D_{11}w + V_7\theta + V_{10}z = -V_8z\theta - (1/2)V_3z^2 - (1/2)V_6\phi^2 - (3/2)V_9\theta^2 - m_{14}z\dot{q} - m_{14}\dot{z}q - m_{15}\theta\dot{q} - m_{15}\dot{\theta}q - m_{25}z\dot{u} - m_{25}\dot{z}u - m_{26}\theta\dot{u} - m_{26}\dot{\theta}u - m_{32}\phi\dot{v} - m_{32}\dot{\phi}v - m_{38}z\dot{w} - m_{38}\dot{z}w - m_{39}\theta\dot{w} - m_{39}\dot{\theta}w - m_{41}\phi\dot{p} - m_{41}\dot{\phi}p - m_{45}\phi\dot{r} - m_{45}\dot{\phi}r - (m_1 - m_7 - m_{22})uw - (m_{24} - m_{39})wq + (m_{26} + m_{37})uq - (m_{10} - m_{16} - m_{44} + I_{xx})pr - (m_{29} - m_{35})vr + (m_{31} + m_{33})vp + [(1/2)m_3 + m_{20}]u^2 + (1/2)m_6v^2 + [(1/2)m_9 - m_{20}]w^2 + [(1/2)m_{12} + m_{42} - I_{xz}]p^2 + (1/2)m_{15}q^2 + [(1/2)m_{18} - m_{42} + I_{xz}]r^2 \tag{36e}$$

$$(I_{zz} + m_{16})\dot{r} + m_{33}\dot{v} + (m_{42} - I_{xz})\dot{p} + D_6r + D_{10}v + D_{12}p = V_{10}z\phi - (V_4 - V_7)\phi\theta - m_{17}z\dot{r} - m_{17}\dot{z}r - m_{18}\theta\dot{r} - m_{18}\dot{\theta}r - m_{27}\phi\dot{u} - m_{27}\dot{\phi}u - m_{34}z\dot{v} - m_{34}\dot{z}v - m_{35}\theta\dot{v} - m_{35}\dot{\theta}v - m_{40}\phi\dot{w} - m_{40}\dot{\phi}w - m_{43}z\dot{p} - m_{43}\dot{z}p - m_{44}\theta\dot{p} - m_{44}\dot{\theta}p - m_{45}\phi\dot{q} - m_{45}\dot{\phi}q + m_{20}vw + (m_{24} + m_{29})vq - (m_{24} + m_{29})up - m_{33}ur - m_{37}wp + (m_{42} - I_{xz})qr + (m_{10} - m_{13} + I_{xx} - I_{yy})pq + (m_1 - m_4)uv \tag{36f}$$

V. Conclusions

The nonlinear equations of motion obtained by assuming Taylor expansions for the hydrodynamic forces and moments can lead to the prediction of unrealistic, self-sustained oscillations in calm water, unless certain restrictions are imposed on the coefficients. These restrictions are in addition to those which result from assuming the ship to be laterally symmetric and from eliminating velocity-acceleration interactions and terms involving accelerations to higher orders. To demonstrate this without involving excessive algebra, we have performed a nonlinear analysis of the motion of a ship which is free to pitch and roll only. We chose this example because, for some ships, these modes are strongly coupled through the nonlinear terms. The analysis provides the restrictions on the coefficients which eliminate the unrealistic oscillations.

As an alternate approach, we propose an energy formulation in which the ship and the sea are considered a single dynamic system. With this approach expansions are assumed for the kinetic and potential energies and the dissipation. The possibility of predicting unrealistic results is eliminated by requiring the kinetic energy and the dissipation to be positive definite for every motion and the potential energy to increase with every displacement from the equilibrium position. The hydrodynamic loads are obtained from the combination of derivatives indicated in Eq. (6). For the example mentioned previously, the equations of motion obtained through the energy approach are essentially the same as those obtained by expanding the moments and forces directly, after the restrictions eliminating the unrealistic oscillations are imposed.

Thus, there are two equivalent approaches to the development of the equations of motion. One can choose either to assume expansions for the forces and moments directly or to assume expansions for the energies and dissipation and then determine the forces and moments from these expansions. The energy approach appears to be much more direct and reliable because of the ease with which one can eliminate the possibility of predicting unrealistic oscillations.

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